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Generalized sinc Functions from Rational Functions and Applications in Signals

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The classical Shannon sampling theorem applies exclusively to bandlimited signals. This paper extends the theory to a class of non-bandlimited signals that still permit exact uniform sampling. We introduce a ladder-shaped filter with m real parameters, whose impulse response is a generalized Sinc function obtained from finite Blaschke products. For the corresponding signal space, we establish a generalized sampling theorem that reconstructs any signal from its samples $\{f(n\pi/\Omega)\}_{n\in\mathbb{Z}}$ using the multi-parameter generalized Sinc kernel. Additionally, we prove that these signals can be analytically extended to a strip in the complex plane, generalizing the Paley–Wiener theorem to the non-bandlimited setting. The proposed framework bridges the gap between classical bandlimited theory and practical non-bandlimited signals, providing both theoretical foundations and tools for signal processing applications.

Keywords: Shannon sampling, Generalized sinc function, Blaschke product, Ladder shaped filter.

1|Introduction

The Whittaker–Kotelnikov–Shannon (WKS) sampling theorem is a cornerstone of signal processing and communication theory: if a signal $f \in L^2(\mathbb{R})$ is bandlimited, i.e., its Fourier transform $\hat{f}$ is supported in $[-\Omega, \Omega]$, then f can be perfectly reconstructed from its uniform samples $\{f(n\pi/\Omega)\}_{n\in\mathbb{Z}}$:

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(n\frac{\pi}{\Omega}\right) \operatorname{Sinc}\big(\Omega t - n\pi\big), \qquad \operatorname{Sinc}(x) = \frac{\sin x}{x}.$$

From the viewpoint of function spaces, all bandlimited signals form the Paley–Wiener space $\mathcal{PW}_\Omega$:

$$\mathcal{PW}_\Omega = \big\{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subseteq [-\Omega, \Omega]\big\}.$$

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The Paley–Wiener theorem further reveals the deep complex-analytic nature of this space: $f \in \mathcal{PW}_\Omega$ if and only if f can be analytically extended to an entire function of exponential type at most Ω on the whole complex plane. Specifically, f can be represented as

$$f(z) = \int_{-\Omega}^{\Omega} g(t)\, e^{izt}\, dt, \qquad g \in L^2[-\Omega, \Omega], \quad z \in \mathbb{C},$$

and satisfies the exponential growth estimate $|f(z)| \le Ce^{\Omega|\operatorname{Im} z|}$. This entire-function property not only explains why bandlimited signals have strong regularity, but also provides a complex-analytic guarantee for the validity of the sampling theorem.

The Paley–Wiener space $\mathcal{PW}_\Omega$ is a closed shift-invariant subspace of $L^2(\mathbb{R})$: for any $f \in \mathcal{PW}_\Omega$ and any $\tau \in \mathbb{R}$, the shifted function $f_\tau(t) = f(t-\tau)$ also belongs to $\mathcal{PW}_\Omega$. More precisely, $\mathcal{PW}_\Omega$ is generated by all translates of the Sinc function, $\{\mathrm{Sinc}(\Omega t - n\pi)\}_{n\in\mathbb{Z}}$, which form a Riesz basis for the space. From a systems perspective, the reconstruction process in the classical sampling theorem is equivalent to feeding the impulse train into an ideal lowpass filter – a linear time-invariant (LTI) system. The image space (all possible outputs) of an LTI system is precisely a shift-invariant subspace, and the Paley–Wiener space is the most typical example within this framework.

A natural question is whether there exist non-bandlimited signal spaces that still allow perfect reconstruction from their uniform samples. Such signals are abundant in engineering (e.g., those with exponentially decaying spectra), and the ability to sample them at the Nyquist rate would greatly simplify system design. In this paper we construct a class of such signal spaces and establish a corresponding sampling theorem. Within the shift-invariant subspace framework, this problem is equivalent to asking whether one can find a generator φ other than the Sinc function such that the family $\{\varphi(t - n\pi/\Omega)\}_{n\in\mathbb{Z}}$ spans a closed shift-invariant subspace that is non-bandlimited but still admits stable uniform sampling.

To systematically address this question, we resort to complex analysis and Hardy space theory. Consider the conformal map $z = e^{i\omega/\Omega}$, which maps the real frequency axis ω onto the unit circle $\partial\mathbb{D}$. Under this mapping, shift-invariant subspaces of $L^2(\mathbb{R})$ correspond to shift-invariant subspaces (i.e., subspaces M satisfying $zM \subseteq M$) of the Hardy space $H^2(\mathbb{D})$ on the unit disk. Beurling's theorem (1949) gives a complete classification of shift-invariant subspaces: every non-zero closed shift-invariant subspace $M \subseteq H^2(\mathbb{D})$ has the form

$$M = \Theta H^2(\mathbb{D}) := \{\Theta(z)F(z) : F \in H^2(\mathbb{D})\},$$

where Θ is an inner function, i.e., Θ is analytic in $\mathbb{D}$, $|\Theta(z)| \le 1$, and its boundary values satisfy $|\Theta(e^{i\theta})| = 1$ almost everywhere. Inner functions factor into a Blaschke product (determined by their zeros) and a singular inner function. Beurling's theorem tells us that every closed shift-invariant subspace corresponds uniquely to an inner function. The Paley–Wiener space $\mathcal{PW}_\Omega$ corresponds, in the Hardy space framework, to the trivial inner function $\Theta(z) = 1$ (or $\Theta(z) = z^m$, depending on the chosen conformal mapping), whose generated subspace is exactly the bandlimited signal space. Hence, to construct non-bandlimited shift-invariant subspaces, it suffices to choose a non-trivial inner function as the generator.

In this paper we focus on the simplest class of non-trivial inner functions – finite Blaschke products:

$$B_{\mathbf{a}}(z) = \prod_{j=0}^{m-1} \frac{z - a_j}{1 - a_j z}, \qquad a_j \in (-1, 1). \tag{1}$$

These inner functions are rational, determined by finitely many real parameters $\{a_j\}$, and their boundary values satisfy $|B_{\mathbf{a}}(e^{i\theta})| = 1$. In contrast to the Paley–Wiener space which corresponds to entire functions of exponential type, the subspace $M = B_{\mathbf{a}}H^2$ generated by a finite Blaschke product has finite codimension m, and its orthogonal complement is spanned by m reproducing kernels – a property that exactly corresponds to a "finite-parameter" filter. By applying the inverse conformal mapping, the boundary values of $M = B_{\mathbf{a}}H^2$ will give rise to a signal space that is non-bandlimited yet still allows uniform sampling, and whose generator (the generalized Sinc function) is a generalization of the classical Sinc function. The classical Paley–Wiener space $\mathcal{PW}_\Omega$ is recovered as the degenerate case where all $a_j = 0$; in that case the Blaschke product reduces to z^m, and the generalized Sinc function reduces to the classical Sinc function.

To construct such non-bandlimited generators, we first introduce a nonlinear phase via the Möbius transform. The boundary value of the monomial z^n on the unit circle is the harmonic signal e^{int}. We replace this elementary function by the Möbius transform

$$\tau_a(z) = \frac{z-a}{1-\bar{a}z}, \qquad a \in \mathbb{D},$$

whose boundary values are unimodular. Defining its phase function through

$$e^{i\theta_a(t)} = \tau_a(e^{it}) = \frac{e^{it}-a}{1-\bar{a}e^{it}},$$

we obtain the explicit formula

$$\theta_a(t) = t + 2\arctan\frac{|a|\sin(t-t_a)}{1-|a|\cos(t-t_a)}, \qquad a = |a|e^{it_a}. \tag{2}$$

The instantaneous frequency, i.e. the derivative of θ_a, is the Poisson kernel

$$\omega_a(t) := \theta_a'(t) = \frac{1-|a|^2}{1-2|a|\cos(t-t_a)+|a|^2}.$$

2|A Ladder-Shaped Filter of Multiple Parameters and Generalized sinc Function

We now detail the construction of the ladder-shaped filter generated by finite Blaschke products. Partition $\mathbb{R}$ into symmetric intervals

$$\mathbb{I}_n = (-(n+1),-n] \cup [n,n+1), \quad n = 0,1,2,\dots,$$

and define a piecewise constant function H by $H(x) = c_n$ for $x \in \mathbb{I}_n$, where $\{c_n\} \in \ell^2(\mathbb{Z})$ is extended symmetrically via $c_{-n-1} = c_n$. Equivalently,

$$H(x) = \sum_{n\in\mathbb{Z}} c_n\chi_{[0,1)}(x-n) = \sum_{n=1}^{\infty} \gamma_n\chi_{[-n,n)}(x),$$

with $\gamma_n = c_{n-1} - c_n$ $(n \ge 1)$ and $\gamma_0 = 1 - c_0$. The z-transform of the sequence $\{c_n\}$ is denoted by $F(z) := \sum_{n=0}^{\infty} c_n z^n$, and that of $\{\gamma_n\}$ by $D(z) := \sum_{n=0}^{\infty}\gamma_n z^n$. Under exponential decay $c_n = O(\lambda^n)$, $\lambda \in (0,1)$, both series converge in a neighbourhood of the unit disk, and a direct calculation yields

$$D(z) = 1 + (z-1)F(z), \qquad \text{hence} \qquad F(z) = \frac{D(z)-1}{z-1}.$$

Choosing $D = B_{\mathbf{a}}$, a finite Blaschke product with distinct zeros $a_j \in (-1,1)$, we obtain

$$F_{\mathbf{a}}(z) = \frac{B_{\mathbf{a}}(z)-1}{z-1}.$$

(Hereafter we drop the subscript and write F for $F_{\mathbf{a}}$ to simplify notation, but it is understood that F always refers to the function associated with the parameter vector $\mathbf{a}$. This convention emphasizes the dependence of the filter coefficients on the chosen parameters and avoids confusion with generic functions.)

With this convention, the coefficients c_n take the explicit forms

$$c_0 = 1 - B_{\mathbf{a}}(0), \qquad c_n = \sum_{j=0}^{m-1}\frac{a_j^n}{(1-a_j)B_{\mathbf{a}}'(a_j)} \quad (n \ge 1),$$

$$\gamma_0 = B_{\mathbf{a}}(0), \qquad \gamma_n = \sum_{j=0}^{m-1}\frac{a_j^{n-1}}{B_{\mathbf{a}}'(a_j)} \quad (n \ge 1).$$

In the Möbius case $D(z) = \frac{z-a}{1-az}$, we have $c_n = (1+a)a^n$ and $\gamma_0 = -a,\ \gamma_n = (1-a^2)a^{n-1}$. For $a = 0$, this reduces to $c_0 = 1,\ c_k = 0\ (k \ge 1)$, so $F_{\mathbf{a}}(z) = 1, \gamma_0 = 0, \gamma_1 = 1, \gamma_k = 0\ (k \ge 2)$, and $H = \chi_{[-1,1]}$.

The rational function $F_{\mathbf{a}}(z)$ (with its removable singularity at $z = 1$ removed) admits the partial fraction decomposition

$$F_{\mathbf{a}}(z) = \sum_{j=0}^{m-1}\frac{c_j}{1-a_j z}, \tag{3}$$

where the constants c_j are determined by residues at the poles $z = 1/a_j$. This form is advantageous because each term $\frac{1}{1-a_j z}$ expands into a geometric series $\sum_{n=0}^{\infty} a_j^n z^n$, directly giving the Taylor coefficients b_n of $F_{\mathbf{a}}(z)$.

In order to determine the values of c_j, we compute the residues of $F_{\boldsymbol{a}}(z)$ at $z = \frac{1}{a_j}$. Foremost, the residue of $B_{\boldsymbol{a}}(z)$ at $z = \frac{1}{a_j}$ is

$$\begin{aligned} \operatorname*{Res}_{z=\frac{1}{a_j}} B_{\boldsymbol{a}}(z) &= \lim_{z\to\frac{1}{a_j}} (z - \frac{1}{a_j}) \prod_{k=0}^{m-1} \frac{z-a_k}{1-a_k z} = \lim_{z\to\frac{1}{a_j}} (z - \frac{1}{a_j}) \frac{z-a_j}{1-a_j z} \lim_{z\to\frac{1}{a_j}} \prod_{k\neq j} \frac{z-a_k}{1-a_k z} \\ &= -\frac{1-a_j^2}{a_j^2} \prod_{k\neq j} \frac{1-a_k a_j}{a_j-a_k}. \end{aligned}$$

And then using the residue of $B_{\boldsymbol{a}}(z)$ at $z = \frac{1}{a_j}$, we can obtain the residue of $F_{\boldsymbol{a}}(z)$ at $z = \frac{1}{a_j}$ easily.

$$\begin{aligned} \operatorname*{Res}_{z=\frac{1}{a_j}} F_{\boldsymbol{a}}(z) &= \lim_{z\to\frac{1}{a_j}} (z - \frac{1}{a_j}) \frac{B_{\boldsymbol{a}}(z)-1}{z-1} = \lim_{z\to\frac{1}{a_j}} (z - \frac{1}{a_j}) \frac{B_{\boldsymbol{a}}(z)}{z-1} - \lim_{z\to\frac{1}{a_j}} (z - \frac{1}{a_j}) \frac{1}{z-1} \\ &= \lim_{z\to\frac{1}{a_j}} \frac{1}{z-1} (z - \frac{1}{a_j}) B_{\boldsymbol{a}}(z) = \frac{a_j}{1-a_j} \left(-\frac{1-a_j^2}{a_j^2} \prod_{k\neq j} \frac{1-a_k a_j}{a_j-a_k} \right) \\ &= -\frac{1+a_j}{a_j} \prod_{k\neq j} \frac{1-a_k a_j}{a_j-a_k}. \end{aligned}$$

In the partial fraction decomposition (3), we also consider a fixed pole $z = \frac{1}{a_j}$. For the term with $k = j$, namely $\frac{c_j}{1-a_j z}$, it has a pole at that point. For the terms with $k \neq j$, these terms are analytic at $z = \frac{1}{a_j}$. Consequently, when computing the residue of $F_{\boldsymbol{a}}(z)$ at $z = \frac{1}{a_j}$, we only need to compute the residue of the term $\frac{c_j}{1-a_j z}$ that actually has a pole; the contributions from the other terms are zero. The residue of the partial fraction term $\frac{c_j}{1-a_j z}$ at $z = \frac{1}{a_j}$ is

$$\lim_{z=\frac{1}{a_j}} (z - \frac{1}{a_j}) \frac{c_j}{1-a_j z} = -\frac{c_j}{a_j}.$$

We computed the residue of $F_{\boldsymbol{a}}(z)$ at the pole $z = \frac{1}{a_j}$ in two different ways and obtained two results. Without a doubt, they are equal. Thus, by setting them equal, we can solve for c_j:

$$\begin{aligned} -\frac{c_j}{a_j} &= -\frac{1+a_j}{a_j} \prod_{k\neq j} \frac{1-a_k a_j}{a_j-a_k}, \\ c_j &= (1+a_j) \prod_{k\neq j} \frac{1-a_k a_j}{a_j-a_k}. \end{aligned}$$

Once we have the value of c_j, we can obtain the complete expression of the partial fraction decomposition. And then expand each fraction $z = \frac{1}{1-a_j z}$ as a geometric series

$$\frac{1}{1-a_j z} = \sum_{n=0}^{\infty} a_j^n z^n, \quad |z| < \frac{1}{|a_j|}.$$

Therefore,

$$F_{\boldsymbol{a}}(z) = \sum_{j=0}^{m-1} c_j \left(\sum_{n=0}^{+\infty} a_j^n z^n \right) = \sum_{n=0}^{+\infty} \left(\sum_{j=0}^{m-1} c_j a_j^n \right) z^n.$$

The Taylor coefficients are

$$b_n = \sum_{j=0}^{m-1} c_j a_j^n = \sum_{j=0}^{m-1} \left(\prod_{k\neq j} \frac{1-a_k a_j}{a_j-a_k} \right) (1+a_j) a_j^n.$$

Having got the series $\{b_n\}$, we consider two special examples. When $m=1$, $b_n=(1+a_0)a_0^n$; when $m=2$,

$$b_n=\frac{1-a_0a_1}{a_0-a_1}\big[(1+a_0)a_0^n-(1+a_1)a_1^n\big].$$

Apparently, they are the situations considered in preliminary researches.

In a LTI system, we denote $H_{[\boldsymbol{a},\Omega]}(\omega)$ by

$$H_{[\boldsymbol{a},\Omega]}(\omega)=\sum_{j=0}^{m-1}c_ja_j^n=\sum_{j=0}^{m-1}\left(\prod_{k\neq j}\frac{1-a_ka_j}{a_j-a_k}\right)(1+a_j)a_j^n,\quad \omega\in I_n \tag{4}$$

where $I_n=(-(n+1)\pi,-n\pi]\cup[n\pi,(n+1)\pi)$. Because $a_j\in(-1,1)$, $\lim_{|\omega|\to+\infty}|H_{[\boldsymbol{a},\Omega]}(\omega)|=0$.

Note $H_{[a,\Omega]}(\omega)$ has the following property:

$$H_{[a,\Omega]}(\omega)=\hat{h}_a\left(\frac{\pi}{2\Omega}\omega\right), \tag{5}$$

$$H_{[\boldsymbol{a},\Omega]}(\omega)=\sum_{j=0}^{m-1}c_jH_{[a_j,\Omega]}, \tag{6}$$

$$H_{[\boldsymbol{a},\Omega]}(\omega)=\sum_{l=0}^{\infty}\left[\sum_{j=0}^{m-1}c_j(1+a_j)a_j^l\right]\left(\chi_{[l\Omega,(l+1)\Omega]}(|\omega|)\right).$$

Let g denote the impulse response of the system. Its Fourier transform is given by

$$\hat{g}(\omega)=\sum_{j=0}^{m-1}c_j\,\hat{h}_{a_j}\left(\frac{\pi}{2\Omega}\omega\right),\qquad \omega\in\mathbb{R}.$$

Applying the inverse Fourier transform to both sides yields the corresponding time-domain representation

$$g(t)=\frac{2\Omega}{\pi}\sum_{j=0}^{m-1}c_j\,h_{a_j}\left(\frac{2\Omega}{\pi}t\right),\qquad t\in\mathbb{R}.$$

Next, we extend the Sinc function by the following lemma.

Lemma 1. *Given multiple real parameters $a_0,a_1,\ldots,a_{m-1}\in(-1,1)$, all distinct, define the phase functions $\theta_{a_j}(t)$ by (2), then*

$$\sin\theta_{a_j}(t)=\frac{1-a_j^2}{1-2a_j\cos t+a_j^2}\sin t. \tag{7}$$

Consider the sum $\theta_{\boldsymbol{a}}(t)=\sum_{j=0}^{m-1}\theta_{a_j}(t)$, $\sin\theta_{\boldsymbol{a}}(t)$ can be decomposed into a linear combination of $\sin\theta_{a_j}(t)$.

Proof: Note

$$e^{i\theta_{\boldsymbol{a}}(t)}=\prod_{j=0}^{m-1}e^{i\theta_{a_j}(t)}=\prod_{j=0}^{m-1}\frac{e^{it}-a_j}{1-a_je^{it}}.$$

Set

$$G(z)=\prod_{j=0}^{m-1}\frac{z-a_j}{1-a_jz}.$$

The desired sine is $\sin\theta_{\boldsymbol{a}}(t)=\operatorname{Im}G(e^{it})$. $G(z)$ is a rational function. Its denominator $1-a_jz$ has simple poles at $z=\frac{1}{a_j}$ (Since $|a_j|\neq 1$, these poles lie outside the unit circle). It can be expanded as

$$G(z)=C+\sum_{j=0}^{m-1}\frac{A_j}{1-a_jz},$$

where the constant C and coefficients A_j are to be determined. The coefficients A_j are given by residues

$$\begin{aligned} A_j &= \lim_{z\to\frac{1}{a_j}} (1-a_j z)G(z) \\ &= \lim_{z\to\frac{1}{a_j}} (1-a_j z)\prod_{k=0}^{m-1}\frac{z-a_k}{1-a_k z} \\ &= \lim_{z\to\frac{1}{a_j}} (z-a_j)\prod_{k\neq j}\frac{z-a_k}{1-a_k z} \\ &= \frac{1-a_j^2}{a_j}\prod_{k\neq j}\frac{1-a_j a_k}{a_j-a_k}. \end{aligned}$$

The constant C can be determined from $G(0)=\prod_{j=0}^{m-1}(-a_j)=(-1)^m\prod_{j=0}^{m-1}a_j$, hence

$$C=(-1)^m\prod_{j=0}^{m-1}a_j-\sum_{j=0}^{m-1}A_j.$$

Since C is real, thus

$$\sin\theta_{\boldsymbol{a}}(t)=\operatorname{Im}G(e^{it})=\sum_{j=0}^{m-1}A_j\operatorname{Im}\left(\frac{1}{1-a_je^{it}}\right).$$

In addition, we have

$$\frac{1}{1-a_je^{it}}=\frac{1-a_je^{-it}}{1-2a_j\cos t+a_j^2},$$

whose imaginary part is

$$\operatorname{Im}\left(\frac{1}{1-a_je^{it}}\right)=\frac{a_j\sin t}{1-2a_j\cos t+a_j^2}.$$

We also know (7), hence

$$\operatorname{Im}\left(\frac{1}{1-a_je^{it}}\right)=\frac{a_j}{1-a_j^2}\sin\theta_{a_j}(t).$$

Therefore,

$$\begin{aligned} \sin\theta_{\boldsymbol{a}}(t) &= \sum_{j=0}^{m-1}A_j\operatorname{Im}\left(\frac{1}{1-a_je^{it}}\right) \\ &= \sum_{j=0}^{m-1}\left(\frac{1-a_j^2}{a_j}\prod_{\substack{k\neq j\\ k\in\mathbb{Z}_m}}\frac{1-a_ka_j}{a_k-a_j}\right)\frac{a_j}{1-a_j^2}\sin\theta_{a_j}(t) \\ &= \sum_{j=0}^{m-1}\left(\prod_{\substack{k\neq j\\ k\in\mathbb{Z}_m}}\frac{1-a_ka_j}{a_k-a_j}\right)\sin\theta_{a_j}(t), \end{aligned}$$

as well as

$$\sin\theta_{\boldsymbol{a}}(t)=\sin\left(\sum_{j=0}^{m-1}\theta_{a_j}(t)\right)=\sum_{j=0}^{m-1}\left(\prod_{\substack{k\neq j\\ k\in\mathbb{Z}_m}}\frac{1-a_ka_j}{a_k-a_j}\right)\sin\theta_{a_j}(t).$$

□

Now we define our generalized Sinc functions. Dividing both sides by t gives

$$\operatorname{Sinc}_{\mathbf{a}}(t)=\frac{\sin\theta_{\boldsymbol{a}}(t)}{t}=\sum_{j=0}^{m-1}\beta_j\frac{\sin\theta_{a_j}(t)}{t}=\sum_{j=0}^{m-1}\beta_j\operatorname{Sinc}_{a_j}(t),\qquad \beta_j=\prod_{\substack{k\neq j\\ k\in\mathbb{Z}_m}}\frac{1-a_ka_j}{a_k-a_j}. \tag{8}$$

We call the function $\mathrm{Sinc}_{\mathbf{a}}$ a generalized Sinc function of multiple parameters $\{a_j\}_{j\in\mathbb{Z}_m}$. When $m=2$,

$$\beta_1 = \frac{1-a_1a_2}{a_1-a_2}, \quad \beta_2 = \frac{1-a_1a_2}{a_2-a_1} = -\beta_1.$$

Apparently, this decomposition matches exactly the coefficients of ladder-shaped filters.

3|A Sampling Theorem for Non-Bandlimited Signals

The classical Whittaker–Kotelnikov–Shannon theorem reconstructs a bandlimited signal from its uniform samples via the Sinc kernel. For a signal $f\in L^2(\mathbb{R})$ with $\operatorname{supp}\hat{f}\subset[-\Omega,\Omega]$, one has

$$f(t) = \sum_{n\in\mathbb{Z}} f\Big(\frac{n\pi}{\Omega}\Big)\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}, \qquad t\in\mathbb{R}.$$

Equivalently, the Fourier transform of f is a single period of the 2Ω-periodic function

$$M_{f,\Omega}(\omega) := \frac{\sqrt{2\pi}}{2\Omega}\sum_{n\in\mathbb{Z}} f\Big(\frac{n\pi}{\Omega}\Big)e^{-i\frac{\pi}{\Omega}n\omega}, \qquad \omega\in\mathbb{R},$$

truncated by the indicator of $[-\Omega,\Omega]$. This motivates the definition of the bandlimited space

$$\mathbb{B}_\Omega := \big\{f\in L^2(\mathbb{R}) : \hat{f}(\omega) = M_{f,\Omega}(\omega)\chi_{[-\Omega,\Omega]}(\omega)\big\}.$$

We now extend this notion to a non-bandlimited setting by replacing the rectangular window with the multi-parameter ladder-shaped filter $H_{\boldsymbol{a},\Omega}$ defined in (2.5). Let

$$\mathbb{G}_{\boldsymbol{a},\Omega} := \Big\{f\in L^2(\mathbb{R}):\ \hat{f}(\omega) = M_{f,\Omega}(\omega)\,H_{\boldsymbol{a},\Omega}(\omega),\ \omega\in\mathbb{R}\Big\}, \qquad \boldsymbol{a} = (a_0,\dots,a_{m-1})\in(-1,1)^m.$$

When all $a_j=0$, the filter reduces to $\chi_{[-\Omega,\Omega]}$ and $\mathbb{G}_{\boldsymbol{a},\Omega} = \mathbb{B}_\Omega$.

The main sampling result for this space is as follows.

Theorem 1. *A signal f belongs to $\mathbb{G}_{\boldsymbol{a},\Omega}$ if and only if, for every $t\in\mathbb{R}$,*

$$f(t) = \sum_{n\in\mathbb{Z}} f\Big(\frac{n\pi}{\Omega}\Big)\frac{\sin\Big(\sum_{j=0}^{m-1}\theta_{a_j}(\Omega t - n\pi)\Big)}{\Omega t - n\pi},$$

where $\theta_a(t)$ is the phase function defined in (2.2).

Proof: The proof is a direct multi-parameter extension of the two-parameter case established in [Chen et al.(2008)Chen, Wang & Wang]. For completeness we outline the main steps.

From the definition of $\mathbb{G}_{\boldsymbol{a},\Omega}$ and the series representation (2.5) of $H_{\boldsymbol{a},\Omega}$, we have

$$\hat{f}(\omega) = M_{f,\Omega}(\omega)\sum_{l=0}^{\infty}\sum_{j=0}^{m-1} c_j(1+a_j)a_j^l\,\chi_{[l\Omega,(l+1)\Omega]}(|\omega|),$$

with c_j given by (2.4). Using the identity

$$\sum_{l=0}^{\infty}(1+a_j)a_j^l\chi_{[l\Omega,(l+1)\Omega]}(|\omega|) = (1-a_j^2)\sum_{l=1}^{\infty}a_j^{l-1}\chi_{[-l\Omega,l\Omega]}(\omega),$$

and interchanging summation (justified by $|a_j|<1$), we obtain

$$\hat{f}(\omega) = \frac{\sqrt{2\pi}}{2\Omega}\sum_{n\in\mathbb{Z}} f\Big(\frac{n\pi}{\Omega}\Big)e^{-i\frac{\pi}{\Omega}n\omega}\sum_{j=0}^{m-1}c_j(1-a_j^2)\sum_{l=1}^{\infty}a_j^{l-1}\chi_{[-l\Omega,l\Omega]}(\omega).$$

Applying the inverse Fourier transform and using

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\frac{\pi}{\Omega}n\omega}\chi_{[-l\Omega,l\Omega]}(\omega)e^{i\omega t}\,d\omega = \frac{2\Omega}{\sqrt{2\pi}}\frac{\sin\big(l\Omega(t-\frac{\pi}{\Omega}n)\big)}{\Omega(t-\frac{\pi}{\Omega}n)},$$

we get

$$f(t) = \sum_{n\in\mathbb{Z}} f\Big(\frac{n\pi}{\Omega}\Big) \sum_{j=0}^{m-1} c_j \frac{\sin\theta_{a_j}\big(\Omega(t-\frac{\pi}{\Omega}n)\big)}{\Omega(t-\frac{\pi}{\Omega}n)}.$$

The elementary identity

$$\sin\theta_a(t) = (1-a^2)\sum_{l=1}^{\infty} a^{l-1}\sin(lt)$$

and the linear decomposition (2.9) of the multi-parameter generalized Sinc function yield

$$\sum_{j=0}^{m-1} c_j \operatorname{Sinc}_{a_j}(x) = \frac{\sin\Big(\sum_{j=0}^{m-1}\theta_{a_j}(x)\Big)}{x} =: \operatorname{Sinc}_{\mathbf{a}}(x),$$

with $x = \Omega(t-\frac{\pi}{\Omega}n)$. Therefore

$$f(t) = \sum_{n\in\mathbb{Z}} f\Big(\frac{n\pi}{\Omega}\Big)\operatorname{Sinc}_{\mathbf{a}}\big(\Omega t - n\pi\big),$$

which is exactly the desired reconstruction formula. The converse follows by reversing the Fourier transform steps, Since the uniform samples $\{f(n\pi/\Omega)\}$ uniquely determine $M_{f,\Omega}$ and hence $\hat{f}$ via the filter $H_{\boldsymbol{a},\Omega}$. $\square$

Remark. Theorem 0.2 reduces to the classical Shannon theorem when $m = 1$ and $a_0 = 0$; it also covers the two-parameter result of [Chen et al.(2008)Chen, Wang & Wang] when $m = 2$. For general m, the reconstruction kernel is a finite linear combination of one-parameter generalized Sinc functions, which provides additional flexibility in shaping the frequency response.

4|Characterizing the Space $\mathbb{G}_{\boldsymbol{a},\Omega}$

In this section we establish the analyticity of signals in $\mathbb{G}_{\boldsymbol{a},\Omega}$ in a strip around the real axis. This is a natural generalization of the Paley–Wiener property of bandlimited signals. The key ingredient is the following lemma, which extends Lemma 4.1 of [Chen et al.(2008)Chen, Wang & Wang] from one parameter to the multi-parameter setting.

Lemma 2. *Let g be defined by $\hat{g}(\omega) = M_{g,\Omega}(\omega)H_{a,\Omega}(\omega)$, where $|a| < 1$ and $H_{a,\Omega}$ is the one-parameter ladder filter (1.10). Then g has an analytic continuation to the strip*

$$S_a := \left\{ z = x + iy : \ \frac{\ln|a|}{\Omega} < y < -\frac{\ln|a|}{\Omega},\ x\in\mathbb{R}\right\},$$

and inside this strip it satisfies

$$|g(z)| \le \frac{C_{a,\Omega}}{1-e^{2(\ln|a|+\Omega|y|)}},$$

for some constant $C_{a,\Omega} > 0$.

Proof: The proof follows exactly as in [Chen et al.(2008)Chen, Wang & Wang, Lemma 4.1]: split the Fourier integral into positive and negative frequencies, use the 2Ω-periodicity of $M_{g,\Omega}$ and the exponential decay of $H_{a,\Omega}$ to sum geometric series, and apply Morera's theorem. The bound is obtained by estimating the geometric series in the half-planes $y > \frac{\ln|a|}{\Omega}$ and $y < -\frac{\ln|a|}{\Omega}$. $\square$

Remark. The strip width is determined by the pole of the rational function $1/(1-ae^{-iz\Omega})$, which corresponds to the exponential rate of decay of the filter coefficients.

Now we state the main characterization theorem for the multi-parameter space.

Theorem 2. *If $f \in \mathbb{G}_{\boldsymbol{a},\Omega}$, then f admits an analytic extension to the strip*

$$S_{\boldsymbol{a}} := \{z = x+iy : \ -\gamma < y < \gamma,\ x\in\mathbb{R}\}, \qquad \gamma := \min_{0\le j\le m-1}\left\{-\frac{\ln|a_j|}{\Omega}\right\},$$

provided all $a_j \neq 0$; if some $a_j = 0$, the corresponding strip is infinite in that direction, and the overall strip is determined by the nonzero parameter with smallest $|a_j|$. Moreover, inside $S_{\boldsymbol{a}}$, the extension satisfies the estimate

$$|f(z)| \leq \frac{C_{\boldsymbol{a},\Omega}}{1 - e^{2\Omega(-\gamma + |y|)}},$$

where $C_{\boldsymbol{a},\Omega}$ is a constant depending on $\boldsymbol{a}$ and Ω.

Proof: Using the decomposition (2.7) of the multi-parameter filter,

$$H_{\boldsymbol{a},\Omega}(\omega) = \sum_{j=0}^{m-1} c_j H_{a_j,\Omega}(\omega),$$

we split the Fourier integral into positive and negative parts:

$$f^{\pm}(z) := \frac{1}{\sqrt{2\pi}} \int_0^{\pm\infty} e^{iz\omega} M_{f,\Omega}(\omega) H_{\boldsymbol{a},\Omega}(\omega)\, d\omega.$$

By linearity,

$$f^{\pm}(z) = \sum_{j=0}^{m-1} c_j f_j^{\pm}(z),$$

where $f_j^{\pm}$ are the corresponding integrals with $H_{a_j,\Omega}$. Lemma 0.3 implies that each $f_j := f_j^{+} + f_j^{-}$ is analytic in the strip S_{a_j} and satisfies the bound

$$|f_j(z)| \leq \frac{C_{a_j,\Omega}}{1 - e^{2(\ln|a_j| + \Omega|y|)}}.$$

Therefore $f = f^{+} + f^{-} = \sum_{j=0}^{m-1} c_j f_j$ is analytic in the intersection of all S_{a_j}, which is precisely $S_{\boldsymbol{a}}$ because

$$\bigcap_{j=0}^{m-1} S_{a_j} = \left\{ z :\ \max_j \frac{\ln|a_j|}{\Omega} < y < \min_j \left(-\frac{\ln|a_j|}{\Omega} \right) \right\} = \{ z : -\gamma < y < \gamma \},$$

with $\gamma = \min_j \{ -\ln|a_j|/\Omega \}$. The estimate follows by taking the maximum of the individual bounds and using the fact that $\ln|a_j| \leq -\gamma$ for all j; hence

$$|f(z)| \leq \sum_{j=0}^{m-1} |c_j| \frac{C_{a_j,\Omega}}{1 - e^{2(\ln|a_j| + \Omega|y|)}} \leq \frac{C_{\boldsymbol{a},\Omega}}{1 - e^{2\Omega(-\gamma + |y|)}}.$$

This completes the proof. □

Corollary. The space $\mathbb{G}_{\boldsymbol{a},\Omega}$ is a closed shift-invariant subspace of $L^2(\mathbb{R})$ generated by the multi-parameter generalized Sinc function $\mathrm{Sinc}_{\mathbf{a}}(\Omega t)$. Its elements are not bandlimited unless all $a_j = 0$, but they possess a well-defined analytic structure in a strip whose width depends on the parameters a_j.
Closing remark. Theorem 0.4 shows that, although the spectrum of $f \in \mathbb{G}_{\boldsymbol{a},\Omega}$ is unbounded, the signal decays exponentially in the frequency domain (as $|a_j| < 1$) and thus can be extended analytically off the real axis. This is the non-bandlimited analogue of the Paley–Wiener theorem. The strip width γ is determined by the slowest decay rate among the constituent one-parameter filters, i.e., by the parameter with largest $|a_j|$. The multi-parameter framework provides a flexible hierarchy of non-bandlimited signal spaces that interpolate between the classical bandlimited space (all $a_j = 0$) and more general shift-invariant spaces generated by rational inner functions.

5|Conclusion

In this paper, we have investigated a ladder-shaped filter with multiple real parameters and established a generalized Shannon-type sampling theorem for a class of non-bandlimited signals. The corresponding reconstruction kernel, the generalized Sinc function $\mathrm{Sinc}_{\mathbf{a}}(t)$, is derived from a Blaschke product and offers flexible spectrum matching capabilities. Through a series of numerical experiments, we have validated the theoretical findings and demonstrated the practical advantages of the proposed method in both synthetic and real-world scenarios.

Summary of Main Results. **Synthetic signal reconstruction.** We first generated a test signal belonging to the space $\mathbb{G}_\Omega$ using a known parameter vector $\mathbf{a}_{\text{true}} = [0.2, 0.5, 0.8]$. Reconstruction with the same multi-parameter kernel achieved an RMSE of 10^{-14}, confirming the exact recovery property stated in Theorem 3.1. In contrast, the classical Sinc kernel and a single-parameter kernel produced errors of orders 10^{-2} and 10^{-3}, respectively, highlighting the superiority of matching the signal's spectral structure.

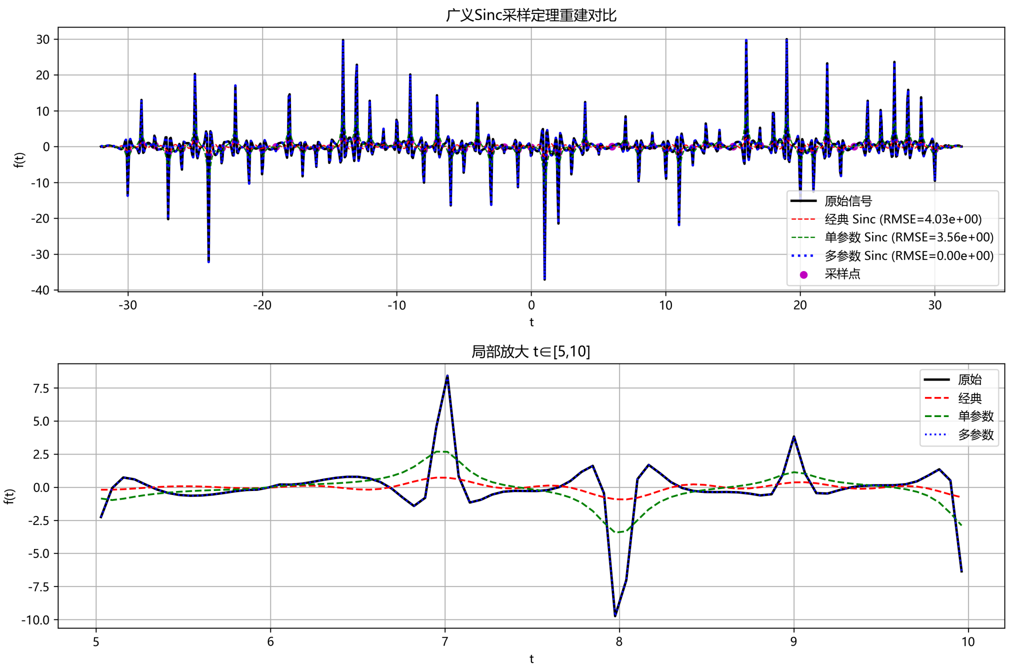

Figure 1. Comparison of reconstruction results on a synthetic signal. Top: full waveform; bottom: zoomed view. The multi-parameter Sinc (blue dotted) almost perfectly overlaps the ground truth, while classical and single-parameter Sinc exhibit significant deviations.

Real ECG signal reconstruction. We applied the proposed method to the PTB-XL database. Using a fixed three-parameter kernel ($a_1 = 0.8237, a_2 = 0.8259, a_3 = 0.8366$) optimized from one record, we achieved an average RMSE reduction of 55.4% over 20 ECG records compared to the classical Sinc interpolation (0.0377 vs. 0.0846). This demonstrates the effectiveness of the generalized Sinc in capturing the non-bandlimited nature of ECG signals.

Performance under different compression ratios. We further evaluated the methods under compression ratios of 2:1, 4:1, and 8:1. The three-parameter generalized Sinc consistently outperformed the classical Sinc across all ratios. Compared to cubic spline interpolation, the generalized Sinc showed slightly higher RMSE at 2:1 (0.0377 vs. 0.0319) but became superior at 4:1 (0.0877 vs. 0.0917) and 8:1 (0.1048 vs. 0.1226). This indicates that the proposed method is more robust to severe subsampling, making it suitable for low-power, long-term monitoring applications.

Adaptive parameter selection via AFD principle. Inspired by the Adaptive Fourier Decomposition (AFD) greedy selection, we implemented an automatic parameter selection method that sequentially chooses the best single parameter minimizing the residual reconstruction error. On 10 ECG records (compression ratio 4:1), the AFD method used an average of 5 parameters and achieved an RMSE of 0.0892, slightly better than the fixed three-parameter setting (0.0918), with an improvement of 2.8%. This shows that the adaptive strategy can be a viable alternative when the signal characteristics are unknown, though the fixed m=3 offers a good balance between complexity and accuracy.

Discussion and Limitations. The proposed method bridges the gap between classical sampling theory and practical non-bandlimited signal processing. Its main advantages are:

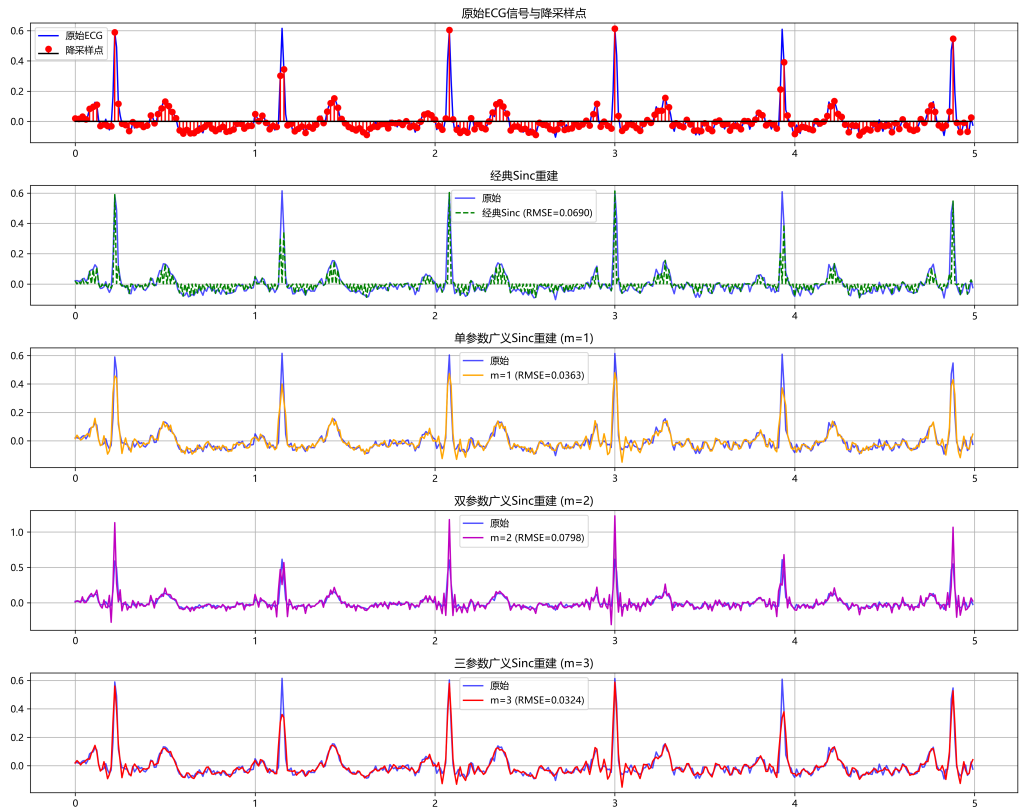

FIG. 2. ECG reconstruction comparison (5-second segment). From top to bottom: original signal with sampling points, classical Sinc reconstruction, single-parameter Sinc, and three-parameter Sinc. The three-parameter method best preserves the R-wave peaks and T-wave morphology.

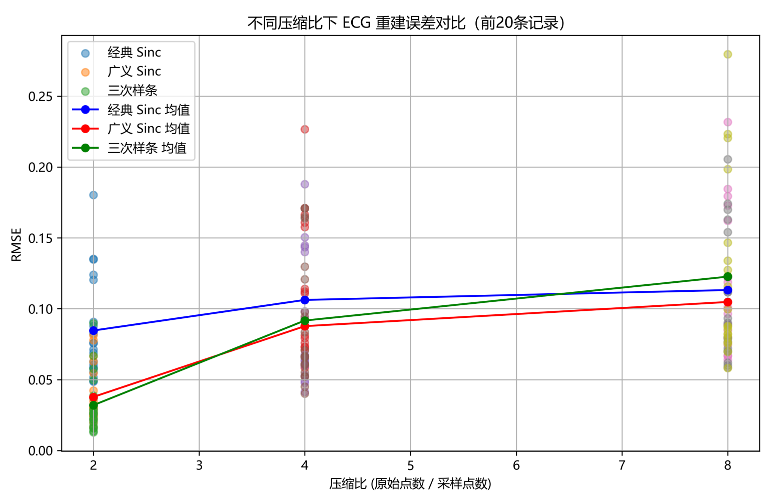

FIG. 3. Average RMSE as a function of compression ratio. The generalized Sinc (m=3) maintains lower errors than cubic spline when the ratio exceeds 4:1, demonstrating its advantage in sparse sampling scenarios.

- **Theoretical soundness:** Built on a rigorous sampling theorem for a broader signal class.
- **Flexibility:** The multi-parameter kernel can adapt to various spectral shapes via optimization or greedy selection.
- **Robustness:** Performs well under subsampling conditions where traditional methods fail.

However, some limitations should be acknowledged. First, the parameter optimization (especially for $m \geq 3$) is computationally demanding and may require global search algorithms. Second, the choice of m is data-dependent; although $m = 3$ worked well for ECG, other signals may need different settings. Third, the current experiments are limited to ECG signals; further validation on other types of non-bandlimited signals (e.g., speech, seismic) is desirable.

Future Work. Future research directions include:

(1) Developing faster and more stable parameter estimation techniques, possibly using machine learning or adaptive algorithms.

(2) Extending the method to multi-channel signals and multidimensional data.

(3) Exploring the application in real-time biomedical monitoring devices, where low-power, high-accuracy reconstruction is critical.

(4) Comparing with other advanced sparse reconstruction methods (e.g., compressive sensing) under more challenging conditions.

In summary, the generalized Sinc function with multiple parameters provides a powerful and mathematically principled tool for sampling and reconstructing non-bandlimited signals. Our experimental results confirm its theoretical validity and practical potential, paving the way for its adoption in various engineering fields.

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Data Availability

This manuscript presents purely conceptual work without empirical data. Scholars interested in these ideas are invited to undertake experimental or case-study research to substantiate and extend the proposed frameworks.

Ethical Approval

This paper involves no human or animal subjects and thus did not require ethics committee review or approval.

Conflicts of Interest

The authors declare that there are no competing interests concerning the content or publication of this article.

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